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Phantom crossing in viable $f(R)$ theories

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We review the equation of state for dark energy in modified gravity theories. In particular, we summarize the generic feature of the phantom divide crossing in the past and future in viable $f(R)$ gravity models.

Keywords: Modified theories of gravity; Equation of state; Dark energy; Cosmology.

To understand the late time acceleration universe ¹, one of the interesting possibilities is to consider a modified gravitational theory, such as $f(R)$ gravity ². To build up a viable $f(R)$ gravity model, one needs to satisfy the following conditions: (a) positivity of the effective gravitational coupling, (b) stability of cosmological perturbations ³, (c) asymptotic behavior to the standard Λ -Cold-Dark-Matter (Λ CDM) model in the large curvature regime, (d) stability of the late-time de Sitter point ⁴, (e) constraints from the equivalence principle, and (f) solar-system constraints ⁵. Several viable models have been constructed in the literature, such as the following popular ones:^{6,7,8,9,10,11,12,13}

model	$f(R)$	Constant parameters
(i) Hu-Sawicki	$R - \frac{c_1 R_{\text{HS}}(R/R_{\text{HS}})^p}{c_2(R/R_{\text{HS}})^p + 1}$	$c_1, c_2, p(>0), R_{\text{HS}}(>0)$
(ii) Starobinsky	$R + \lambda R_{\text{S}} \left[\left(1 + \frac{R^2}{R_{\text{S}}^2} \right)^{-n} - 1 \right]$	$\lambda(>0), n(>0), R_{\text{S}}$
(iii) Tsujikawa	$R - \mu R_{\text{T}} \tanh\left(\frac{R}{R_{\text{T}}}\right)$	$\mu(>0), R_{\text{T}}(>0)$
(iv) Exponential	$R - \beta R_{\text{E}} (1 - e^{-R/R_{\text{E}}})$	β, R_{E}

Recently, the cosmological observational data ¹⁴ seems to indicate the crossing of the phantom divide $w_{\text{DE}} = -1$ of the equation of state for dark energy in the near past. To understand such a crossing, many attempts have been made. The most

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noticeable one is to use a phantom field with a negative kinetic energy term¹⁵. Clearly, it suffers a serious problem as it is not stable at the quantum level. On the other hand, the crossing of the phantom divide can also be realized in the above viable $f(R)$ models^{6,11,12,13,16} without violating any stability conditions. This is probably the most peculiar character of the modified gravitational models. Other $f(R)$ gravity models with realizing a crossing¹⁷ as well as multiple crossings¹⁸ of the phantom boundary have also been examined.

In this talk, we would like to review equation of state in $f(R)$ gravity. In particular, we show that the viable $f(R)$ models generally exhibit the crossings of the phantom divide in the past as well as future^{12,13}.

The action of $f(R)$ gravity with matter is given by

$$I = \int d^4x \sqrt{-g} \frac{f(R)}{2\kappa^2} + I_{\text{matter}}(g_{\mu\nu}, \Upsilon_{\text{matter}}), \quad (1)$$

where g is the determinant of the metric tensor $g_{\mu\nu}$, I_{matter} is the action of matter which is assumed to be minimally coupled to gravity, i.e., the action I is written in the Jordan frame, and Υ_{matter} denotes matter fields. Here, we use the standard metric formalism. By taking the variation of the action in Eq. (1) with respect to $g_{\mu\nu}$, one obtains²

$$FG_{\mu\nu} = \kappa^2 T_{\mu\nu}^{(\text{matter})} - \frac{1}{2} g_{\mu\nu} (FR - f) + \nabla_\mu \nabla_\nu F - g_{\mu\nu} \square F, \quad (2)$$

where $G_{\mu\nu} = R_{\mu\nu} - (1/2)g_{\mu\nu}R$ is the Einstein tensor, $F(R) \equiv df(R)/dR$, ∇_μ is the covariant derivative operator associated with $g_{\mu\nu}$, $\square \equiv g^{\mu\nu}\nabla_\mu\nabla_\nu$ is the covariant d'Alembertian for a scalar field, and $T_{\mu\nu}^{(\text{matter})}$ is the contribution to the energy-momentum tensor from all perfect fluids of matter.

From Eq. (2), we obtain the following gravitational field equations:

$$\begin{aligned} 3FH^2 &= \kappa^2 \rho_M + \frac{1}{2} (FR - f) - 3H\dot{F}, \\ -2F\dot{H} &= \kappa^2 (\rho_M + P_M) + \ddot{F} - H\dot{F}, \end{aligned} \quad (3)$$

where $H = \dot{a}/a$ is the Hubble parameter, the dot denotes the time derivative of $\partial/\partial t$, and ρ_M and P_M are the energy density and pressure of all perfect fluids of matter, respectively.

The equation of state for dark energy is given by

$$w_{\text{DE}} \equiv P_{\text{DE}}/\rho_{\text{DE}}, \quad (4)$$

where

$$\begin{aligned} \rho_{\text{DE}} &= \frac{1}{\kappa^2} \left[\frac{1}{2} (FR - f) - 3H\dot{F} + 3(1 - F)H^2 \right], \\ P_{\text{DE}} &= \frac{1}{\kappa^2} \left[-\frac{1}{2} (FR - f) + \ddot{F} + 2H\dot{F} - (1 - F)(2\dot{H} + 3H^2) \right]. \end{aligned} \quad (5)$$

In Figs. 1, 2 and 3, we depict the evolution of w_{DE} , future evolutions of $1 + w_{\text{DE}}$, and $\tilde{H} \equiv \bar{H} - \bar{H}_f$ with $\bar{H} \equiv H/H_0$ and $\bar{H}_f \equiv H(z = -1)/H_0$, as functions of the redshift $z \equiv 1/a - 1$ in (i) Hu-Sawicki model for $p = 1$, $c_1 = 2$ and $c_2 = 1$, (ii) Starobinsky model for $n = 2$ and $\lambda = 1.5$, (iii) Tsujikawa model for $\mu = 1$ and (iv) the exponential gravity model for $\beta = 1.8$, respectively, and the subscript ‘f’ denotes the value at the final stage $z = -1$. Note that the present time is $z = 0$ and the future is $-1 \leq z < 0$. The parameters used for each model in Figs. 1–3 are the viable ones^{19,20}. Several remarks are as follows: (a) the qualitative results do not strongly depend on the values of the parameters in each model; (b) we have studied the Appleby-Battye model²¹, which is also a viable $f(R)$ model, and we have found that the numerical results are similar to those in the Starobinsky model of (ii) as expected.

We note that the present values of $w_{\text{DE}}(z = 0)$ are -0.92, -0.97, -0.92 and -0.93

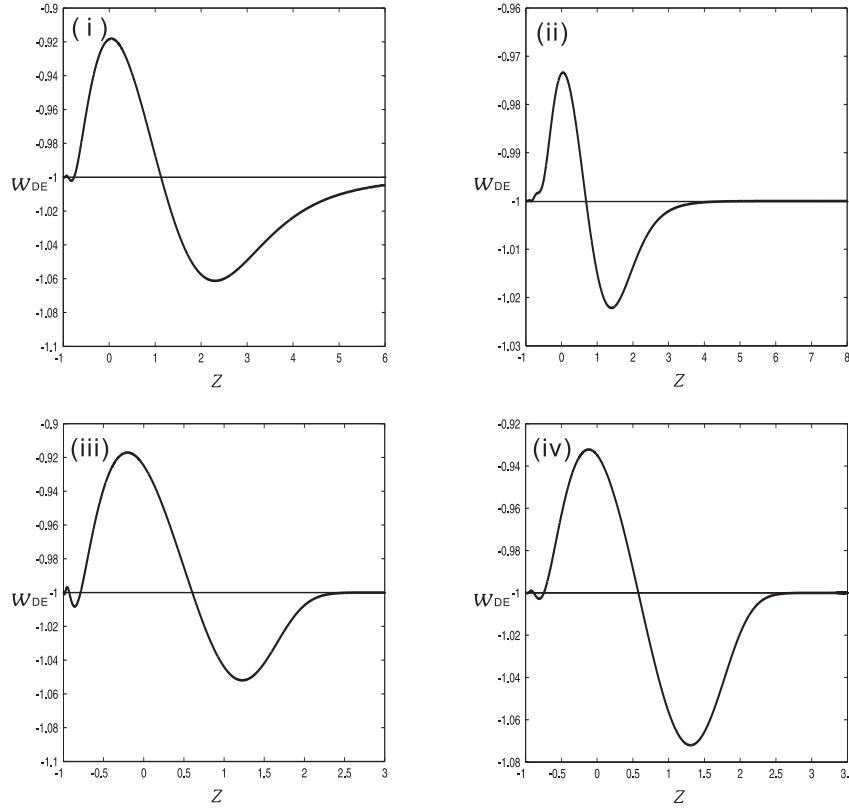
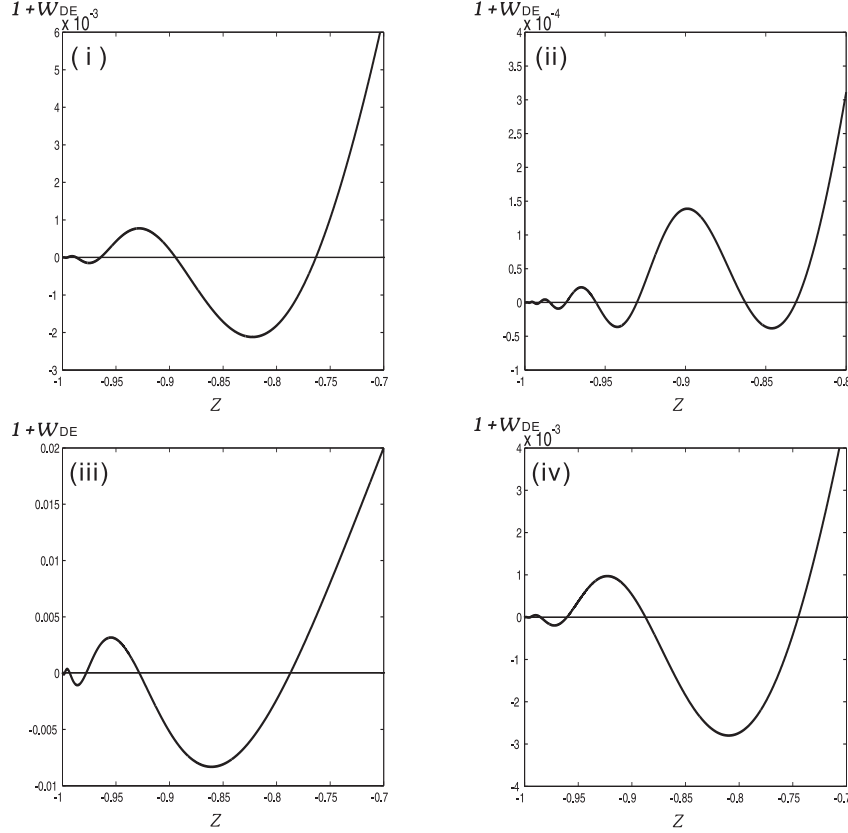


Fig. 1. Evolutions of the equation of state w_{DE} as functions of the redshift z in (i) Hu-Sawicki model for $p = 1$, $c_1 = 2$ and $c_2 = 1$, (ii) Starobinsky model for $n = 2$ and $\lambda = 1.5$, (iii) Tsujikawa model for $\mu = 1$ and (iv) the exponential gravity model for $\beta = 1.8$, respectively, where the thin solid lines show $w_{\text{DE}} = -1$ (cosmological constant).

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Fig. 2. Future evolutions of $1 + w_{\text{DE}}$ as functions of the redshift z . Legend is the same as Fig. 1.

for the models of (i)–(iv), respectively. These values satisfy the present observational constraints²². Moreover, a dimensionless quantity $H^2 / (\kappa^2 \rho_{\text{m}}^{(0)} / 3)$ can be determined through the numerical calculations, where $\rho_{\text{m}}^{(0)}$ is the energy density of non-relativistic matter at the present time. If we use the observational data on the current density parameter of non-relativistic matter $\Omega_{\text{m}}^{(0)} \equiv \rho_{\text{m}}^{(0)} / \rho_{\text{crit}}^{(0)} = 0.26$ with $\rho_{\text{crit}}^{(0)} = 3H_0^2 / \kappa^2$ ²², we find that the present value of the Hubble parameter $H_0 = H(z = 0)$ is $H_0 = 71 \text{ km/s/Mpc}$ ²² for all the models of (i)–(iv). Furthermore, $\bar{H}_{\text{f}} = 0.80, 0.85, 0.78$ and 0.81 for the models of (i)–(iv), respectively, where $\bar{H}_{\text{f}} \equiv H(z = -1) / H_0$.

It is clear from Figs. 1–3 that in the future ($-1 \leq z \lesssim -0.74$), the crossings of the phantom divide are the generic feature for all the existing viable $f(R)$ models. By writing the first future crossing of the phantom divide and the first sign change of \dot{H} from negative to positive as $z = z_{\text{cross}}$ and $z = z_{\text{p}}$, respectively, we find that $(z_{\text{cross}}, z_{\text{p}})_{\alpha} = (-0.76, -0.82)_i, (-0.83, -0.98)_{ii}, (-0.79, -0.80)_{iii}$ and $(-0.74, -0.80)_{iv}$, where the subscript α represents the α th viable model. The values

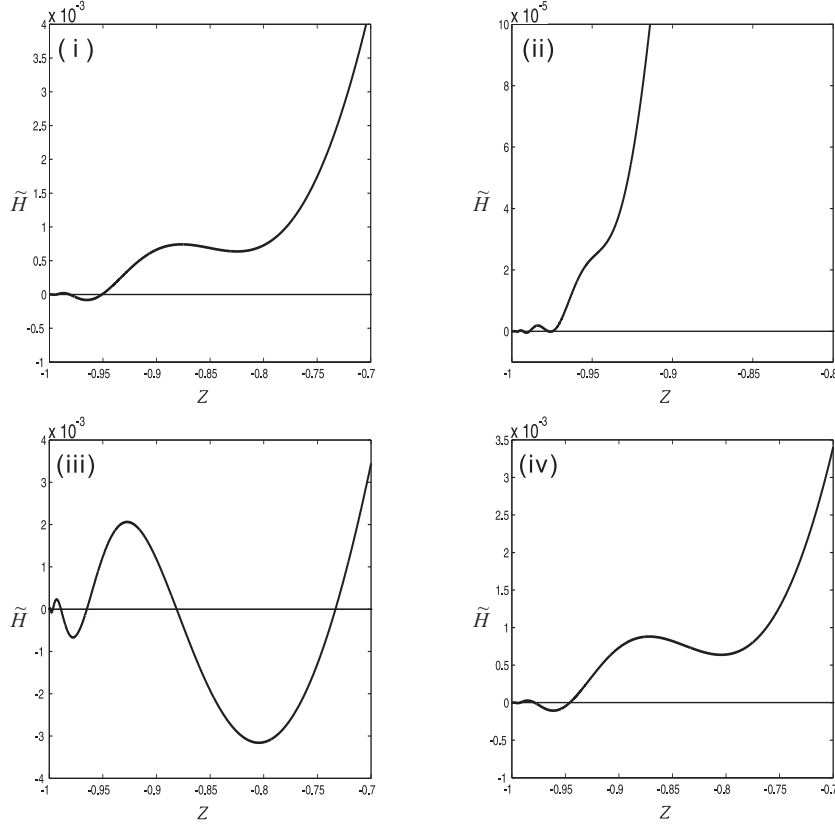


Fig. 3. Future evolutions of $\tilde{H} \equiv \bar{H} - \bar{H}_f$ with $\bar{H} \equiv H/H_0$ and $\bar{H}_f \equiv H(z = -1)/H_0$ as functions of the redshift z . Legend is the same as Fig. 1.

of the ratio $\Xi \equiv \Omega_m/\Omega_{DE}$ at $z = z_{\text{cross}}$ and $z = z_p$ are $(\Xi(z = z_{\text{cross}}), \Xi(z = z_p))_\alpha = (5.2 \times 10^{-3}, 2.1 \times 10^{-3})_i, (1.7 \times 10^{-3}, 4.8 \times 10^{-6})_{ii}, (4.1 \times 10^{-3}, 3.1 \times 10^{-3})_{iii}$ and $(6.2 \times 10^{-3}, 2.8 \times 10^{-3})_{iv}$, where $\Omega_{DE} \equiv \rho_{DE}/\rho_{\text{crit}}^{(0)}$ and $\Omega_m \equiv \rho_m/\rho_{\text{crit}}^{(0)}$ are the density parameters of dark energy and non-relativistic matter (cold dark matter and baryon), respectively. As z decreases ($-1 \leq z \lesssim -0.90$), dark energy becomes much more dominant over non-relativistic matter ($\Xi = \Omega_m/\Omega_{DE} \lesssim 10^{-5}$). As a result, one has $w_{DE} \approx w_{\text{eff}} \equiv -1 - 2\dot{H}/(3H^2) = P_{\text{tot}}/\rho_{\text{tot}}$, where w_{eff} is the effective equation of state for the universe, and $\rho_{\text{tot}} \equiv \rho_{DE} + \rho_m + \rho_r$ and $P_{\text{tot}} \equiv P_{DE} + P_r$ are the total energy density and pressure of the universe, respectively. Here, $\rho_{m(r)}$ and P_r are the energy density of non-relativistic matter (radiation) and the pressure of radiation, respectively. The physical reason why the crossing of the phantom divide appears in the farther future ($-1 \leq z \lesssim -0.90$) is that the sign of \dot{H} changes from negative to positive due to the dominance of dark energy over non-relativistic matter. As $w_{DE} \approx w_{\text{eff}}$ in the farther future, w_{DE} oscillates around the phantom divide line $w_{DE} = -1$ because the sign of \dot{H} changes and consequently multiple crossings can

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be realized.

Finally, we mention that in our numerical calculations, we have taken the initial conditions of $z_0 = 8.0, 8.0, 3.0$ and 3.5 for the models of (i)–(iv) at $z = z_0$, respectively, so that $RF'(z = z_0) \sim 10^{-13}$ with $F' = dF/dR$, to ensure that they can be all close enough to the Λ CDM model with $RF' = 0$.

In this talk, we have explored the past and future evolutions of w_{DE} in the viable $f(R)$ gravity models and explicitly shown that the crossings of the phantom divide are the generic feature in these models. We have demonstrated that in the future the sign of \dot{H} changes from negative to positive due to the dominance of dark energy over non-relativistic matter. This is a common physical phenomena to the existing viable $f(R)$ models and thus it is one of the peculiar properties of $f(R)$ gravity models characterizing the deviation from the Λ CDM model.

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References

1. For a review, see E. J. Copeland, M. Sami and S. Tsujikawa, *Int. J. Mod. Phys. D* **15**, 1753 (2006).
2. For recent reviews, see T. P. Sotiriou and V. Faraoni, *Rev. Mod. Phys.* **82**, 451 (2010); A. De Felice and S. Tsujikawa, arXiv:1002.4928 [gr-qc].
3. S. Nojiri and S. D. Odintsov, *Phys. Rev. D* **68**, 123512 (2003); A. D. Dolgov and M. Kawasaki, *Phys. Lett. B* **573**, 1 (2003); V. Faraoni, *Phys. Rev. D* **74**, 104017 (2006); Y. S. Song, W. Hu and I. Sawicki, *ibid.* **75**, 044004 (2007).
4. V. Muller, H. J. Schmidt and A. A. Starobinsky, *Phys. Lett. B* **202**, 198 (1988); V. Faraoni and S. Nadeau, *Phys. Rev. D* **72**, 124005 (2005); L. Amendola, R. Gannouji, D. Polarski and S. Tsujikawa, *ibid.* **75**, 083504 (2007).
5. T. Chiba, *Phys. Lett. B* **575**, 1 (2003); T. Chiba, T. L. Smith and A. L. Erickcek, *Phys. Rev. D* **75**, 124014 (2007).
6. W. Hu and I. Sawicki, *Phys. Rev. D* **76**, 064004 (2007).
7. A. A. Starobinsky, *JETP Lett.* **86**, 157 (2007).
8. S. Tsujikawa, *Phys. Rev. D* **77**, 023507 (2008).
9. P. Zhang, *Phys. Rev. D* **73**, 123504 (2006); B. Li and M. C. Chu, *ibid.* **74**, 104010 (2006); R. Bean, D. Bernat, L. Pogosian, A. Silvestri and M. Trodden, *ibid.* **75**, 064020 (2007); P. J. Zhang, *ibid.* **76**, 024007 (2007); P. Zhang, M. Liguori, R. Bean and S. Dodelson, *Phys. Rev. Lett.* **99**, 141302 (2007); G. Calcagni and G. Nardelli, arXiv:1004.5144 [hep-th].
10. G. Cognola, E. Elizalde, S. Nojiri, S. D. Odintsov, L. Sebastiani and S. Zerbini, *Phys. Rev. D* **77**, 046009 (2008).
11. E. V. Linder, *Phys. Rev. D* **80**, 123528 (2009).
12. K. Bamba, C. Q. Geng and C. C. Lee, *JCAP* **1008**, 021 (2010) [arXiv:1005.4574 [astro-ph.CO]].
13. K. Bamba, C. Q. Geng and C. C. Lee, *JCAP* **1011**, 001 (2010) [arXiv:1007.0482 [astro-ph.CO]].

14. U. Alam, V. Sahni and A. A. Starobinsky, JCAP **0406**, 008 (2004); **0702**, 011 (2007); S. Nesseris and L. Perivolaropoulos, *ibid.* **0701**, 018 (2007); P. U. Wu and H. W. Yu, Phys. Lett. B **643**, 315 (2006); H. K. Jassal, J. S. Bagla and T. Padmanabhan, arXiv:astro-ph/0601389.
15. R. R. Caldwell, Phys. Lett. B **545**, 23 (2002); B. Feng, X. L. Wang and X. M. Zhang, *ibid.* **607**, 35 (2005); Z. K. Guo, Y. S. Piao, X. M. Zhang and Y. Z. Zhang, *ibid.* **608**, 177 (2005).
16. M. Martinelli, A. Melchiorri and L. Amendola, Phys. Rev. D **79**, 123516 (2009); H. Motohashi, A. A. Starobinsky and J. Yokoyama, arXiv:1002.1141 [astro-ph.CO]; arXiv:1005.1171 [astro-ph.CO].
17. M. C. B. Abdalla, S. Nojiri and S. D. Odintsov, Class. Quant. Grav. **22**, L35 (2005); I. H. Brevik, Int. J. Mod. Phys. D **15**, 767 (2006); Gen. Rel. Grav. **38**, 1317 (2006); L. Amendola and S. Tsujikawa, Phys. Lett. B **660**, 125 (2008); K. Bamba, C. Q. Geng, S. Nojiri and S. D. Odintsov, Phys. Rev. D **79**, 083014 (2009); Mod. Phys. Lett. A **25**, 900 (2010); K. Bamba and C. Q. Geng, Phys. Lett. B **679**, 282 (2009); K. Bamba, arXiv:0904.2655 [gr-qc]; K. Bamba, S. Nojiri and S. D. Odintsov, JCAP **0810**, 045 (2008); Y. Bisabr, arXiv:1005.5679 [gr-qc].
18. K. Bamba and C. Q. Geng, Prog. Theor. Phys. **122**, 1267 (2009).
19. S. Capozziello and S. Tsujikawa, Phys. Rev. D **77**, 107501 (2008); A. Ali, R. Gannouji, M. Sami and A. A. Sen, *ibid.* **81**, 104029 (2010).
20. S. Tsujikawa, R. Gannouji, B. Moraes and D. Polarski, Phys. Rev. D **80**, 084044 (2009).
21. S. A. Appleby and R. A. Battye, Phys. Lett. B **654**, 7 (2007); S. Appleby, R. Battye and A. Starobinsky, JCAP **1006**, 005 (2010).
22. E. Komatsu *et al.*, arXiv:1001.4538 [astro-ph.CO].